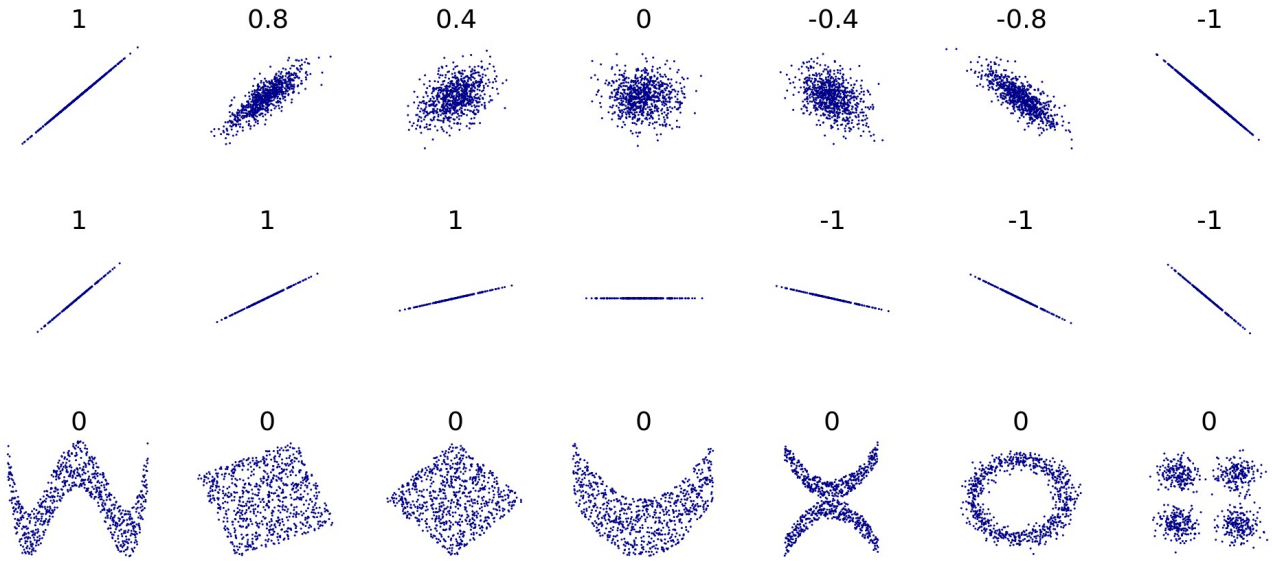
# Pearson Correlations on Complex Networks

Michele Coscia

NERDS Meeting
Jan 19th, 2022

IT UNIVERSITY OF CPH

# Background



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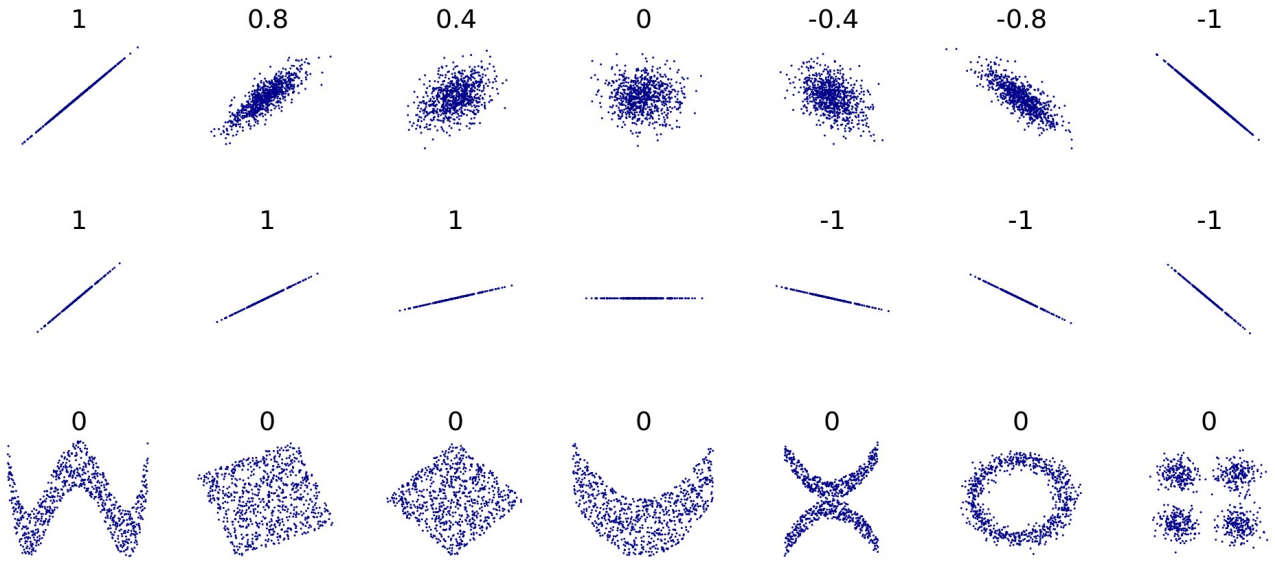


- Correlation = normalized covariance

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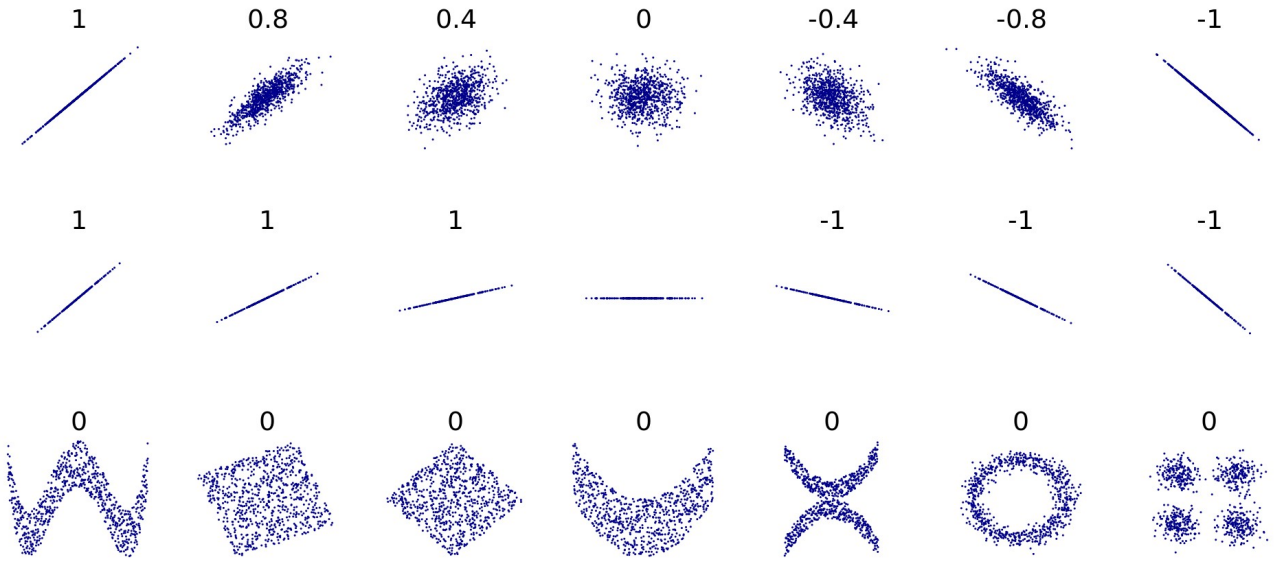
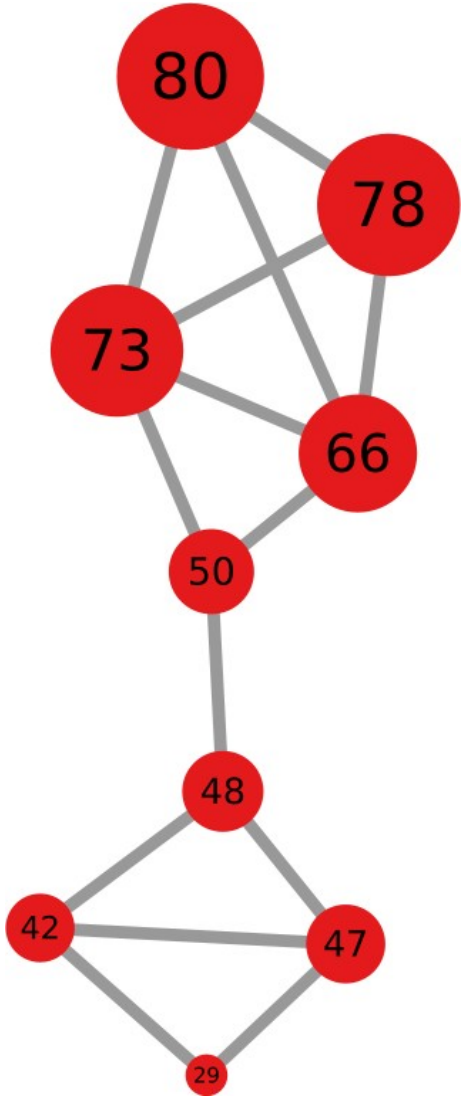


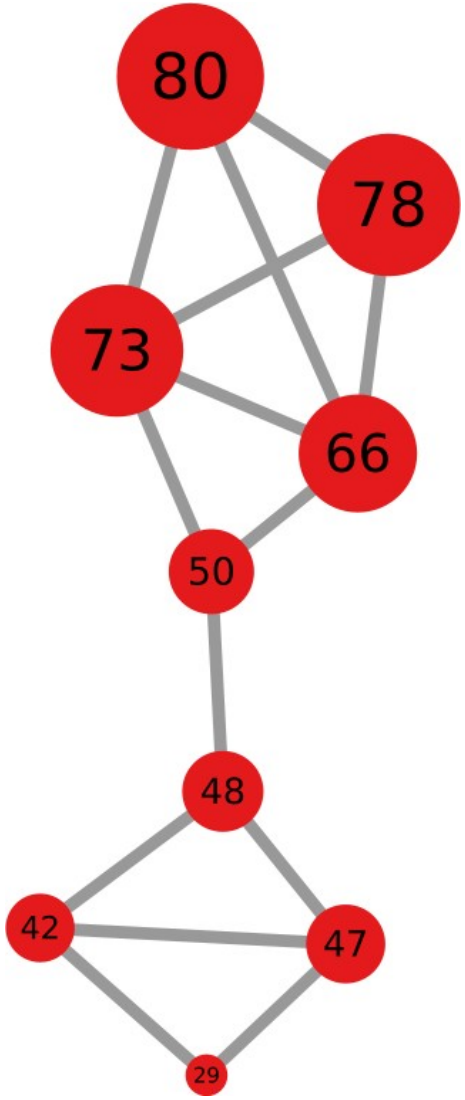

- Correlation = normalized covariance
- How well a linear equation describes the relationship between X and Y

# Motivation



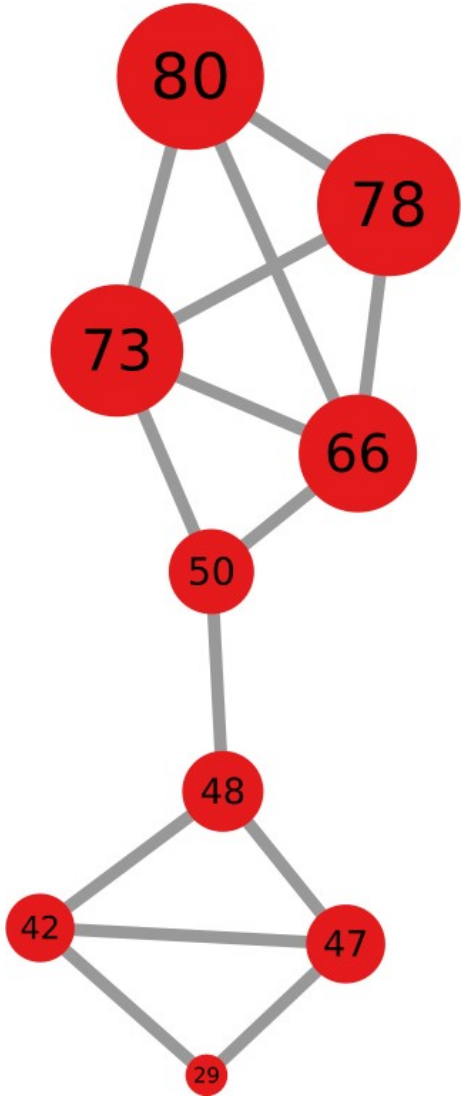
# Motivation

- Represent node properties as numeric vectors



# Motivation

- Represent node properties as numeric vectors
- Many ways in which a network structure influences their values



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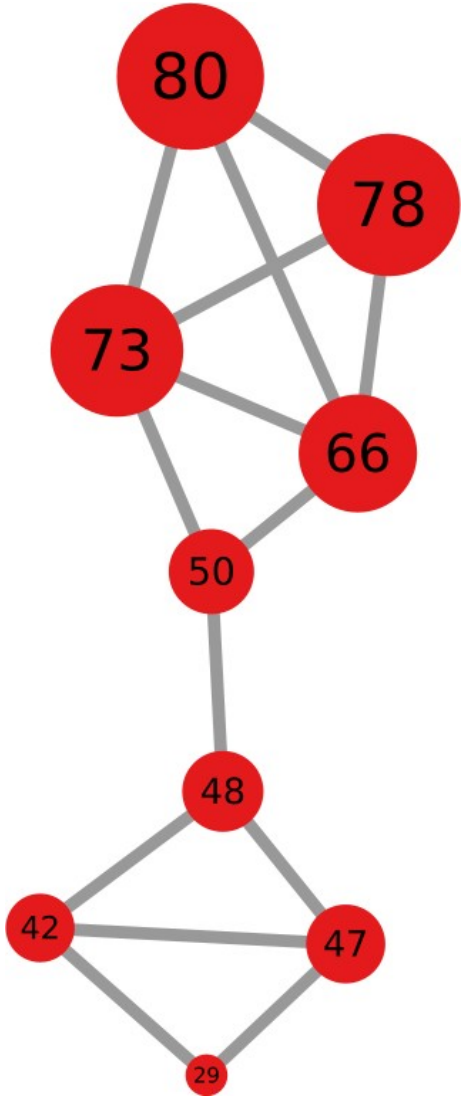
- E.g.: SI/SIS/SIR, homphily/heterophily, etc



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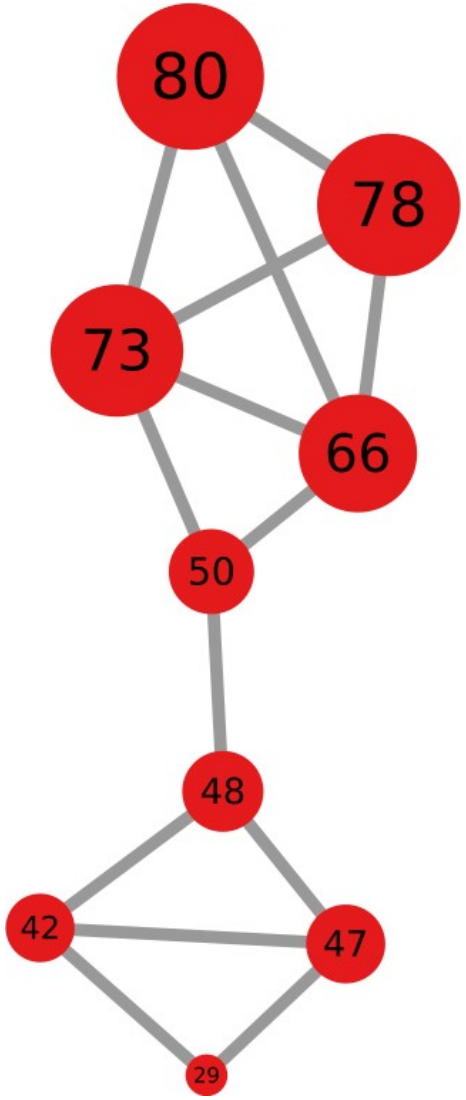
**~ 0.74**



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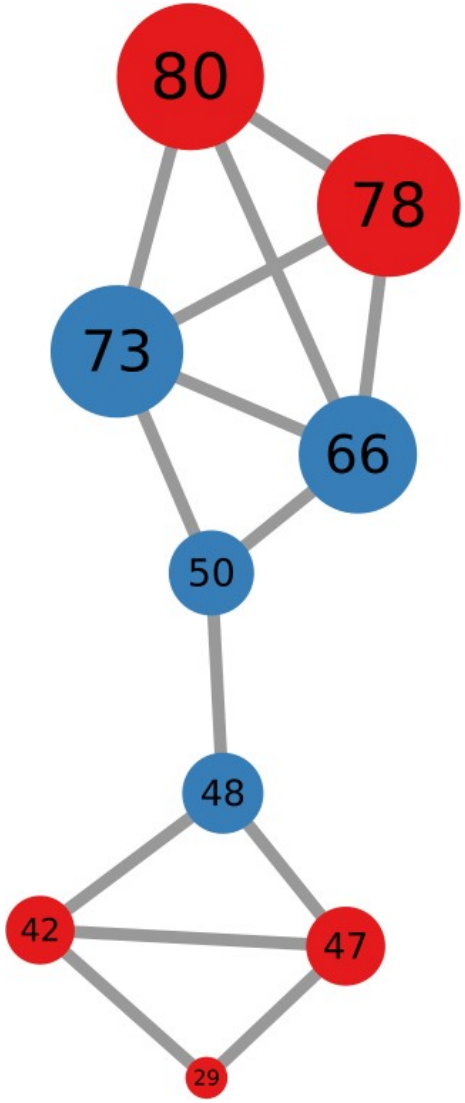

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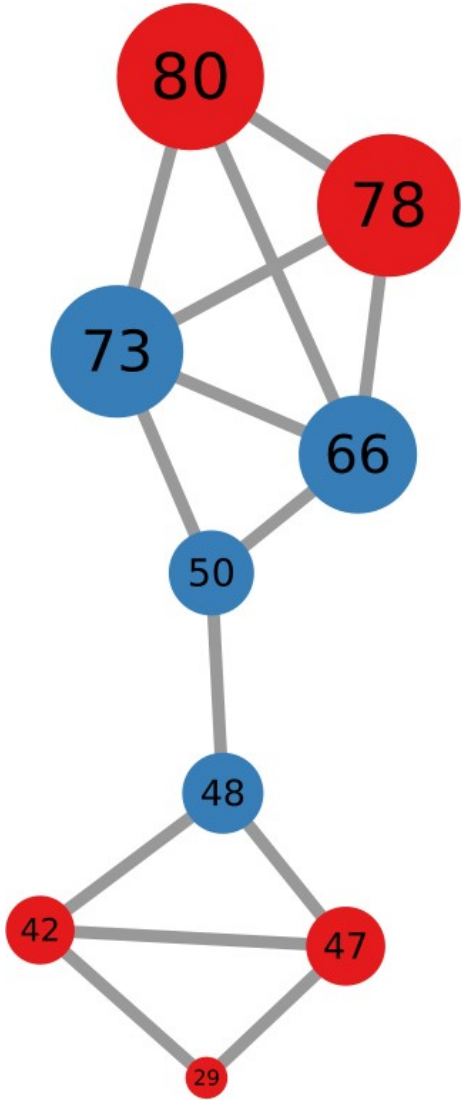
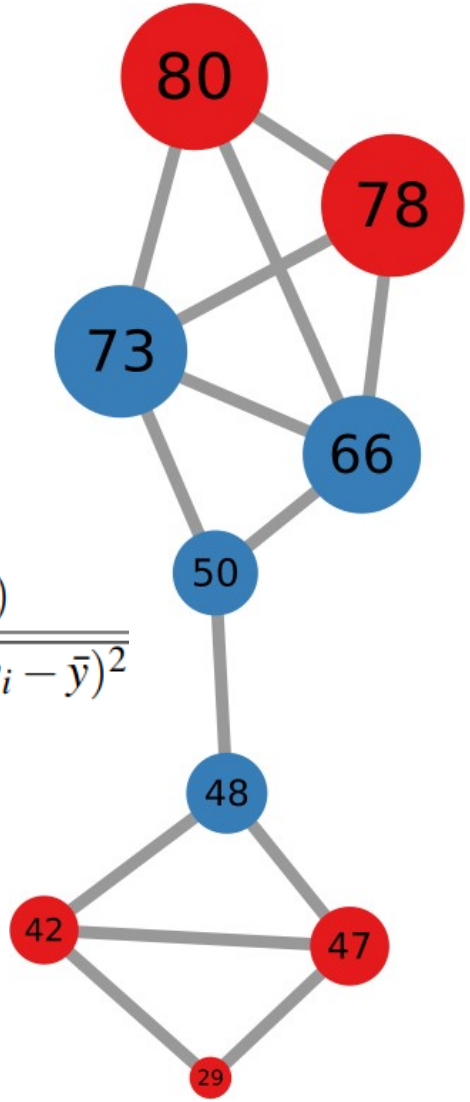
- What if we have more than one variable?



# Motivation

- What if we have more than one variable?
- Pearson doesn't see the network

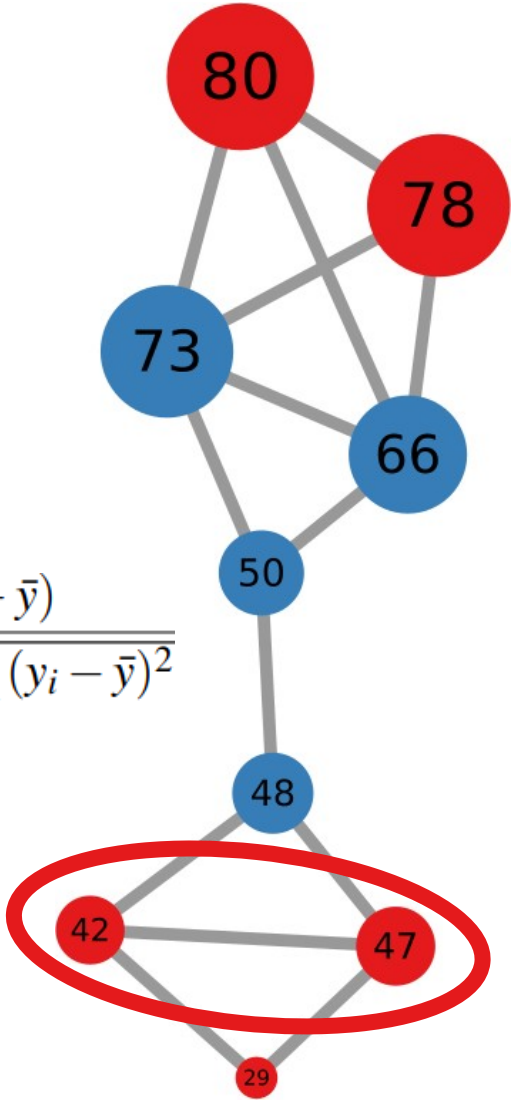
$$\rho_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$



# Motivation

- What if we have more than one variable?
- Pearson doesn't see the network
- Assortativity ignores higher-order relationships

$$\rho_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$



# Research Question

How can we define a Pearson Correlation Coefficient that takes into account the network structure?

# Pearson

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# Pearson

**Covariance of X and Y**

$$\rho_{x,y} = \frac{\boxed{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

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StDev of X

# Pearson

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StDev of X StDev of Y

- Max when Cov(x,y) = SD(x) * SD(y)

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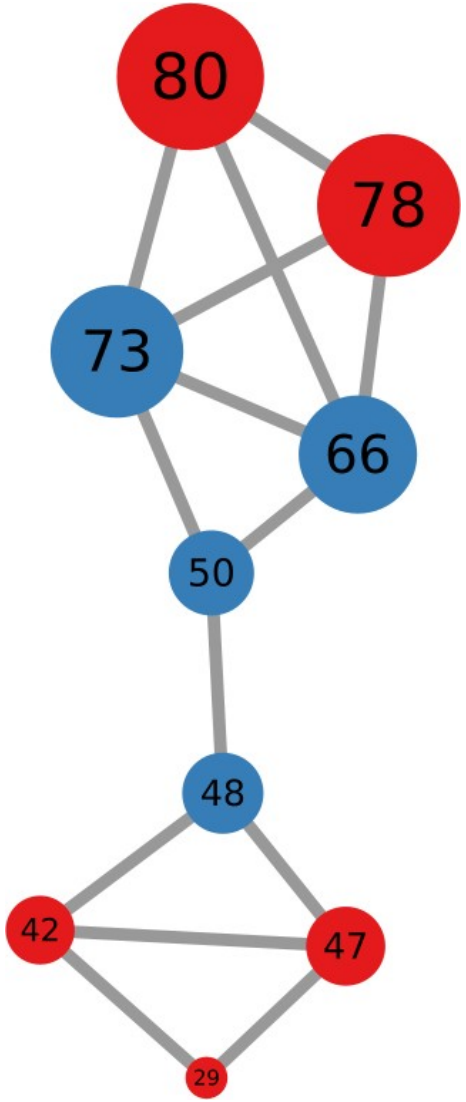
StDev of X StDev of Y

- Max when Cov(x,y) = SD(x) * SD(y)
- Each change in x is proportional to a change in y
- Example: Cov(x,x) = SD(x) * SD(x) = Var(x)

# Pearson & Networks

$$\rho_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$



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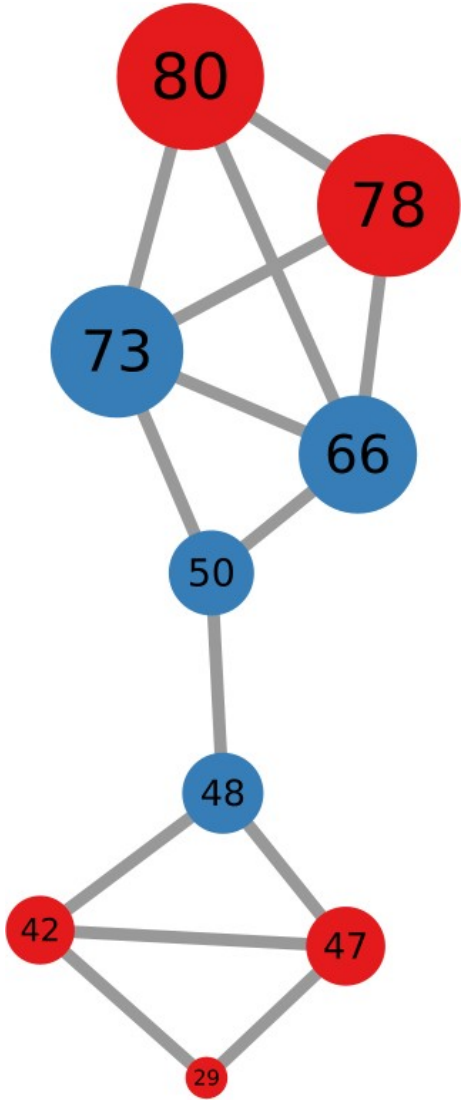

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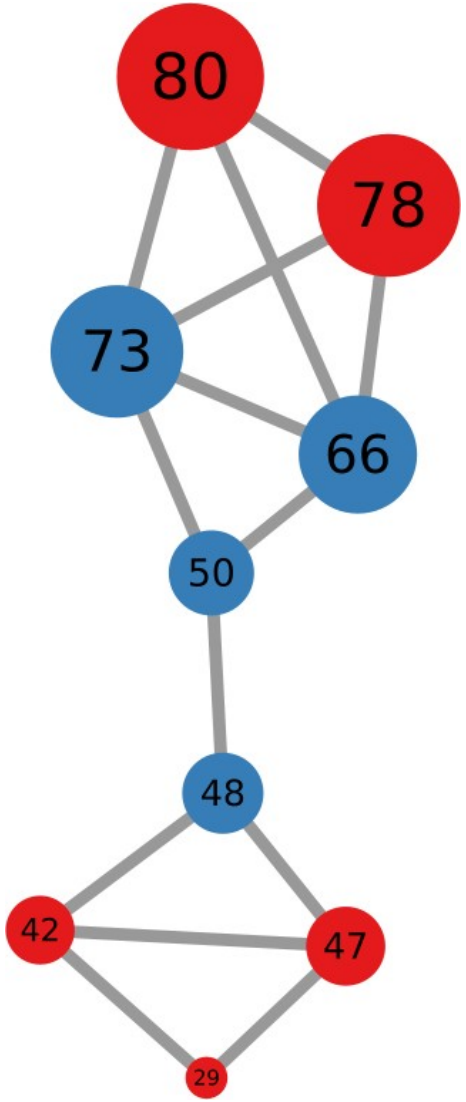
| | |
|---|---|
| 23 | 0.4 |
| 21 | 0.4 |
| 16 | -0.5 |
| 9 | -0.5 |
| -7 | -0.5 |
| -9 | -0.5 |
| -10 | 0.4 |
| -15 | 0.4 |
| -28 | 0.4 |



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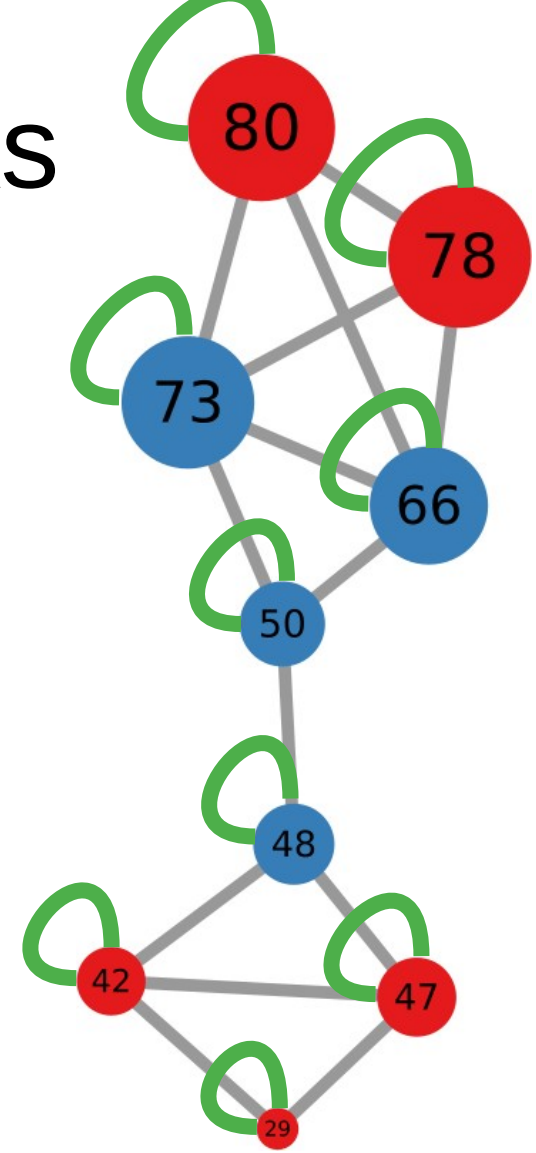

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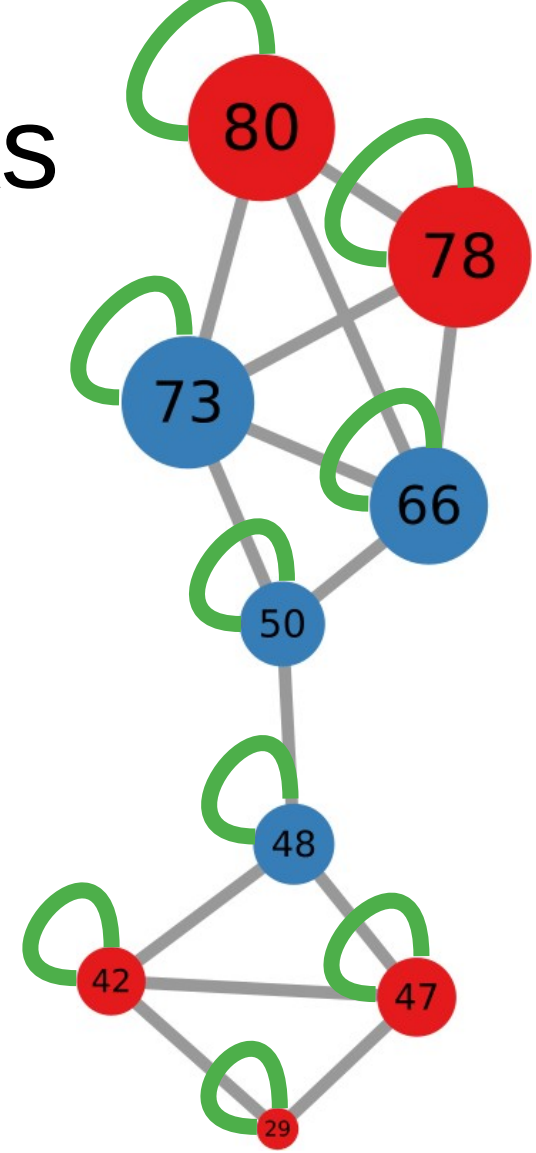
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |



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**sum(10, 9, -9, -5, 4, 5, -4, -7, -12)**

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|---|---|---|---|---|---|---|---|---|---|
| **23** | 10 | 10 | -13 | -13 | -13 | -13 | 10 | 10 | 10 |
| **21** | 9 | 9 | -12 | -12 | -12 | -12 | 9 | 9 | 9 |
| **16** | 7 | 7 | -9 | -9 | -9 | -9 | 7 | 7 | 7 |
| **9** | 4 | 4 | -5 | -5 | -5 | -5 | 4 | 4 | 4 |
| **-7** | 3 | -3 | 4 | 4 | 4 | 4 | -3 | -3 | -3 |
| **-9** | 4 | -4 | 5 | 5 | 5 | 5 | -4 | -4 | -4 |
| **-10** | 4 | -4 | 6 | 6 | 6 | 6 | -4 | -4 | -4 |
| **-15** | 7 | -7 | 8 | 8 | 8 | 8 | -7 | -7 | -7 |
| **-28** | -12 | -12 | 16 | 16 | 16 | 16 | -12 | -12 | -12 |

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Outer

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$$\sigma_x = \sqrt{\mathrm{sum}(I \times (\hat{x} \otimes \hat{x}))}$$

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- My choice: $W = e^{-kL}$
- L = shortest path length
- k = damping factor
- Self loop = 1, exponential decay with distance

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$$\rho_{x,y} = \frac{\mathrm{sum}(I \times (\hat{x} \otimes \hat{y}))}{\sigma_x \sigma_y} \longrightarrow \rho_{x,y,G} = \frac{\mathrm{sum}(W \times (\hat{x} \otimes \hat{y}))}{\sigma_{x,W}\, \sigma_{y,W}}$$

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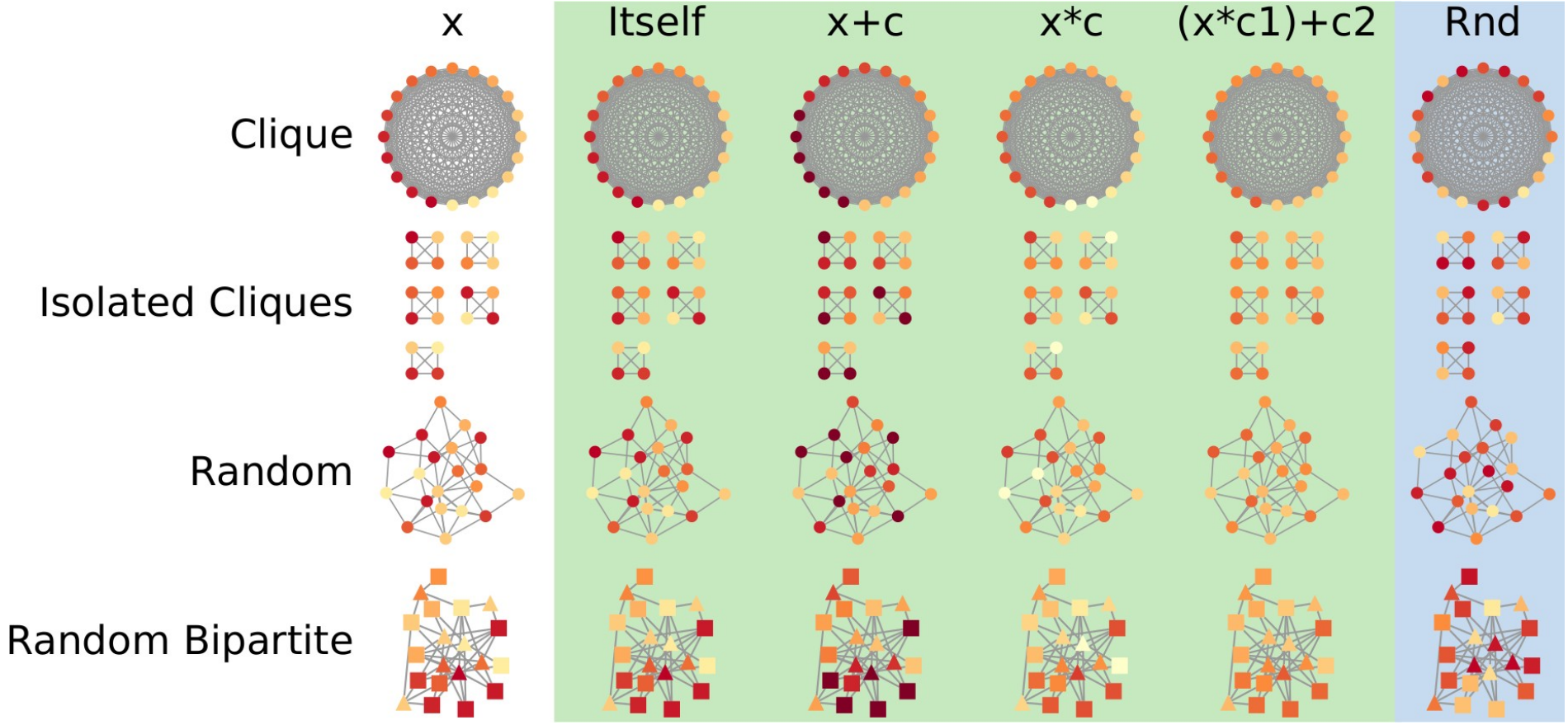
$$\sigma_x = \sqrt{\mathrm{sum}(I \times (\hat{x} \otimes \hat{x}))} \longrightarrow \sigma_{x,W} = \sqrt{\mathrm{sum}(W \times (\hat{x} \otimes \hat{x}))}$$

# Validation: Expected Behavior

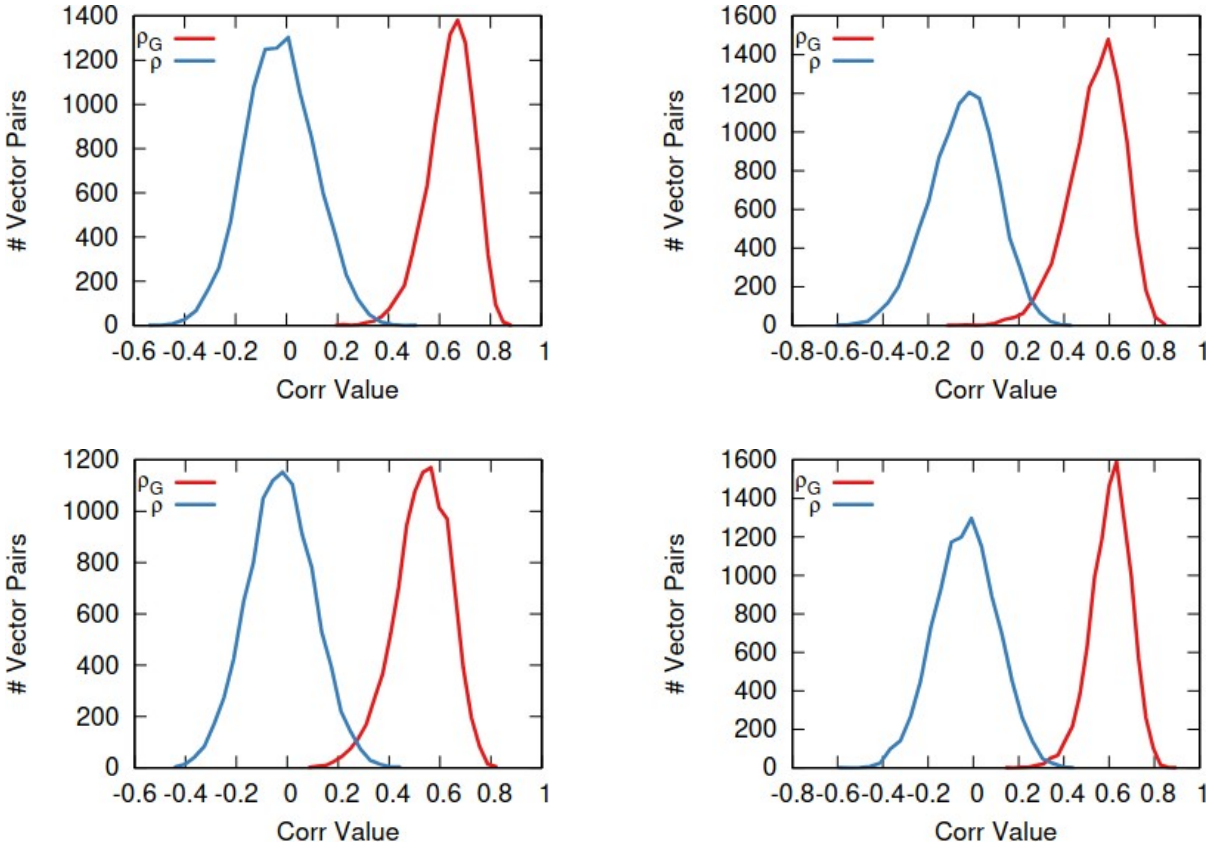


# Validation: Difference with Baseline

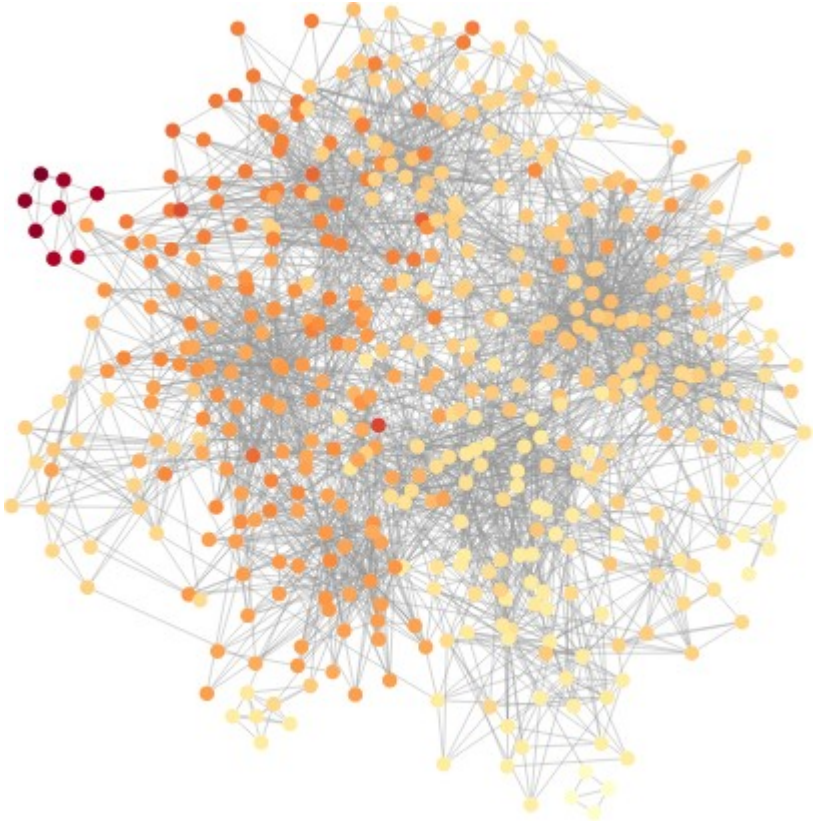
$$y_v = \sum_{u \in N_v} x_u / |N_v|$$

# Validation: Difference with Baseline

$$y_v = \sum_{u \in N_v} x_u / |N_v|$$

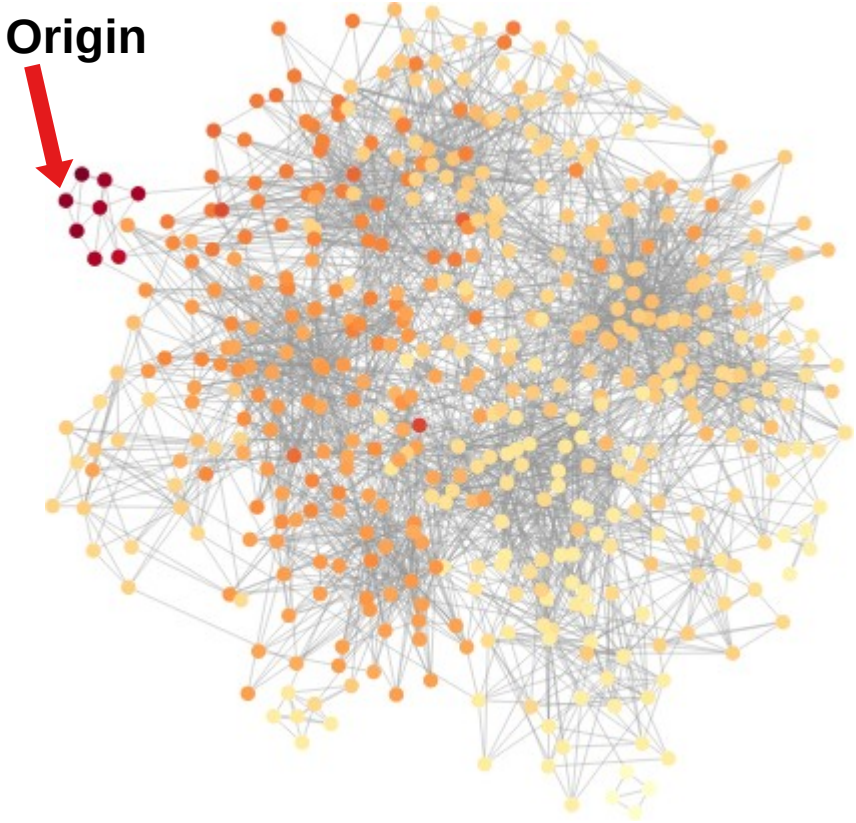


# Validation: Diffusion Processes



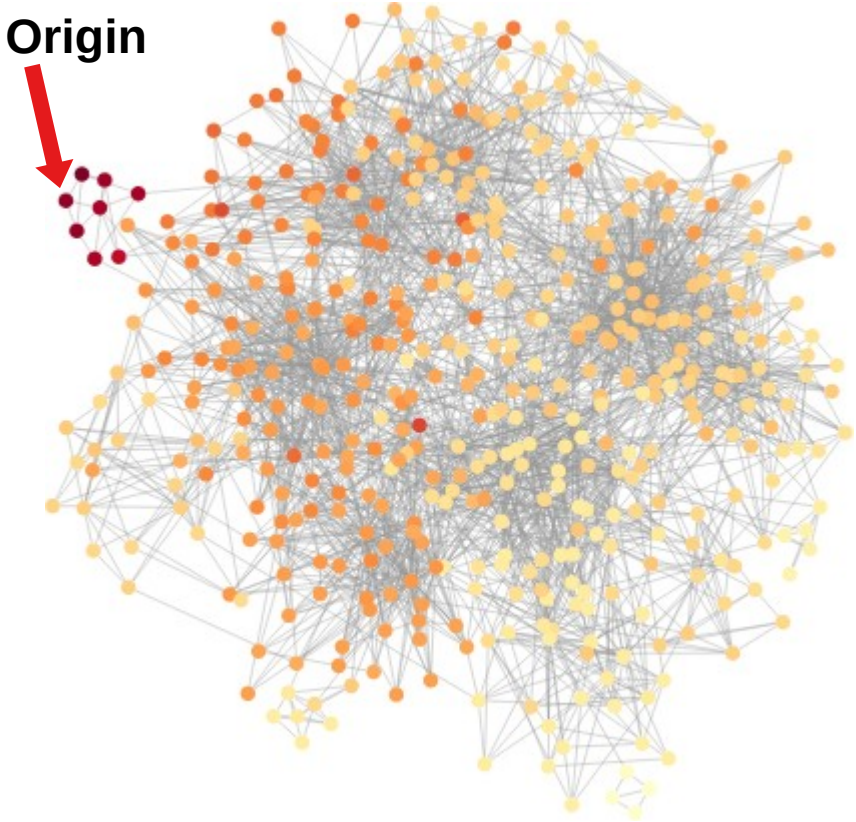
**x**

# Validation: Diffusion Processes

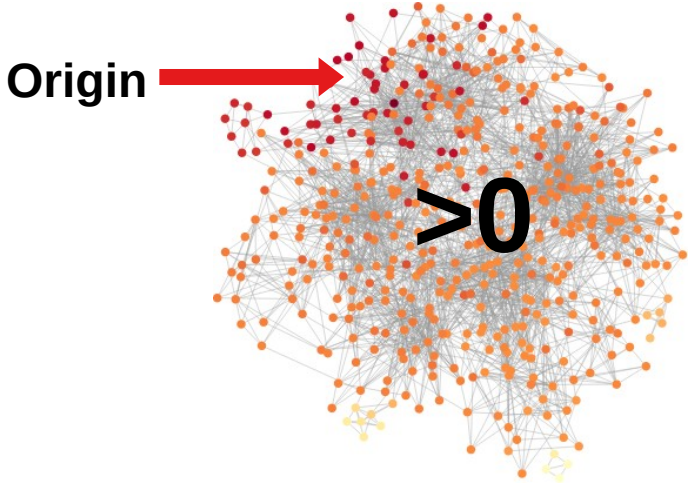



**X**

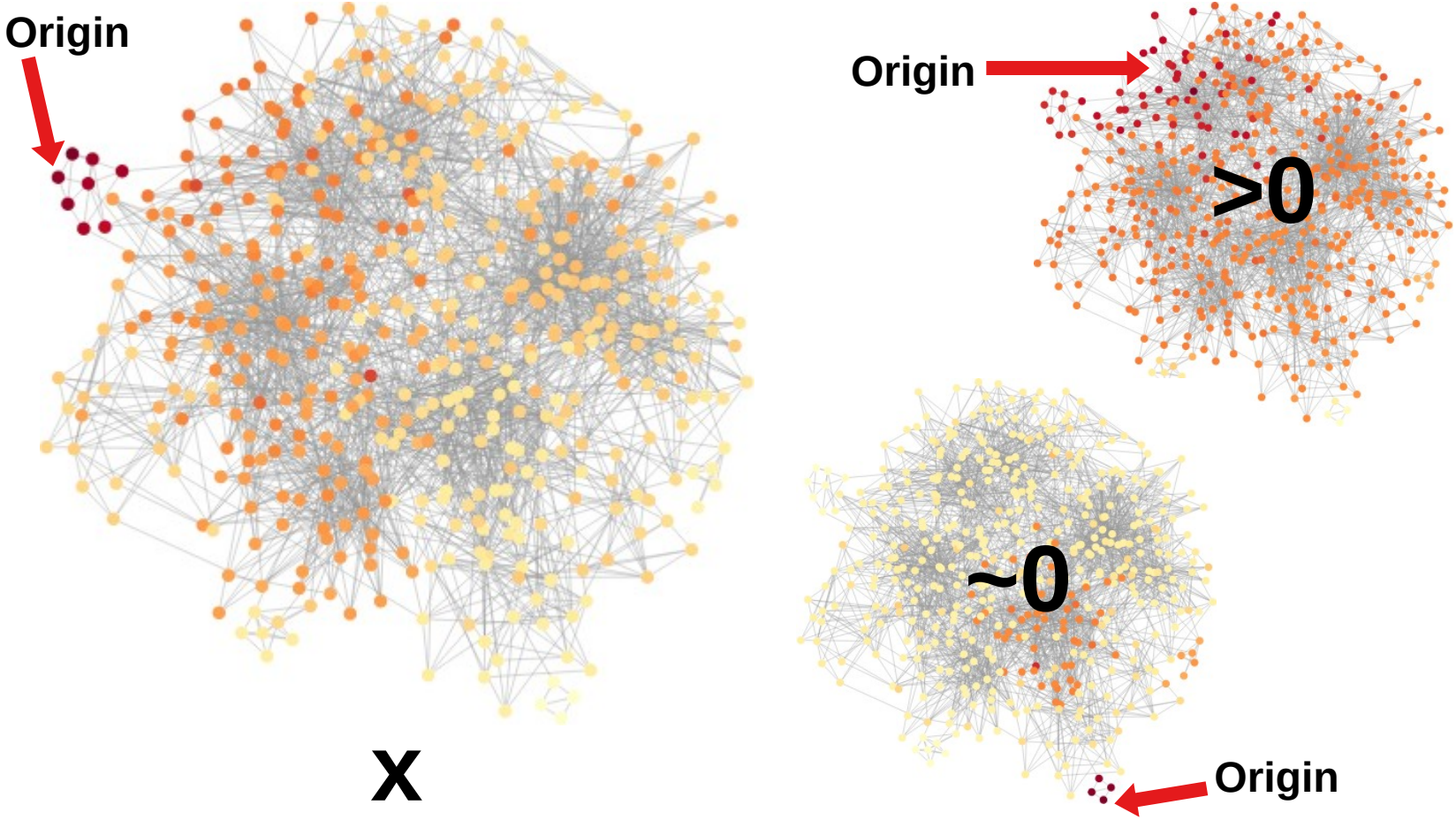
# Validation: Diffusion Processes



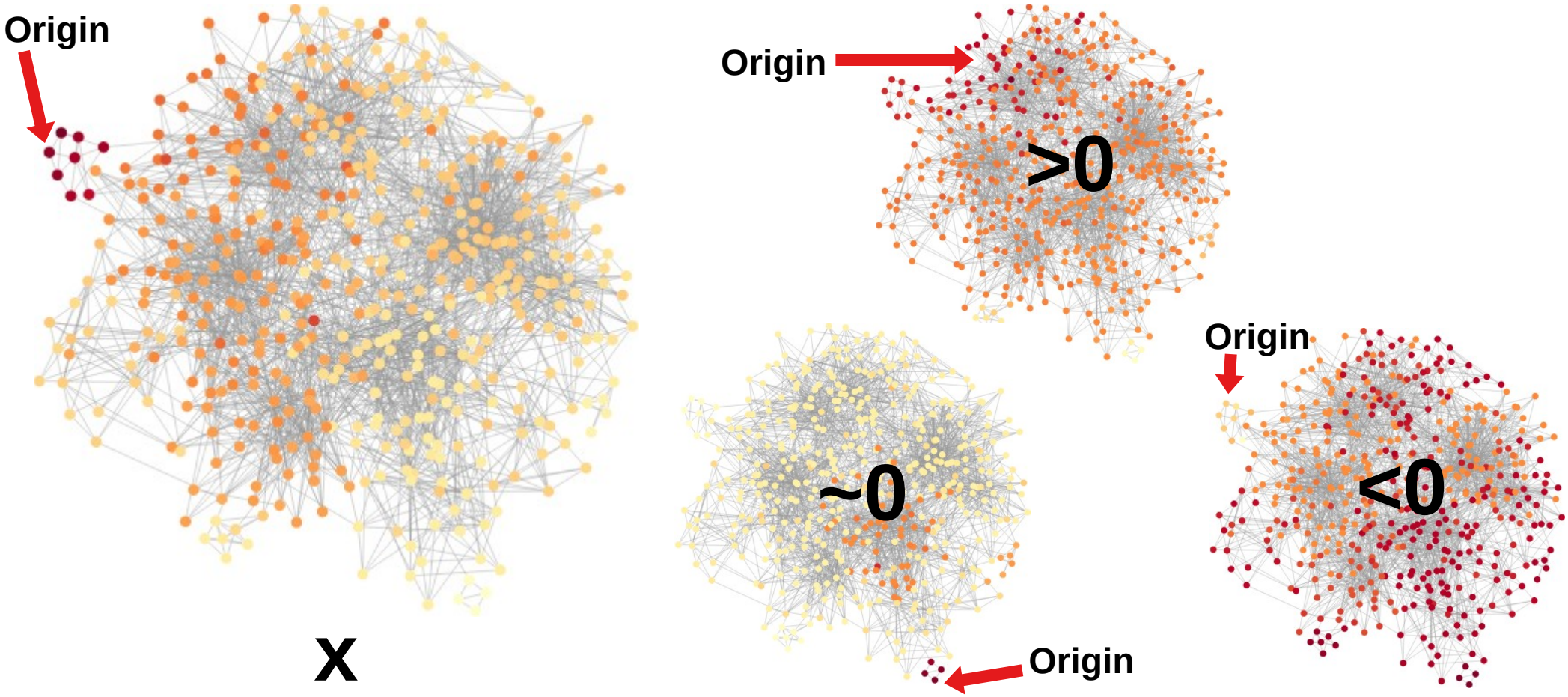

x



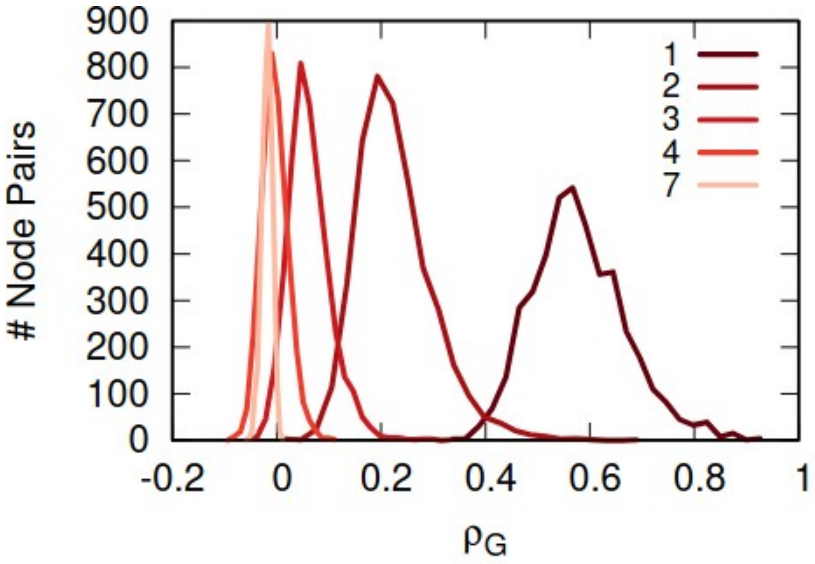
# Validation: Diffusion Processes



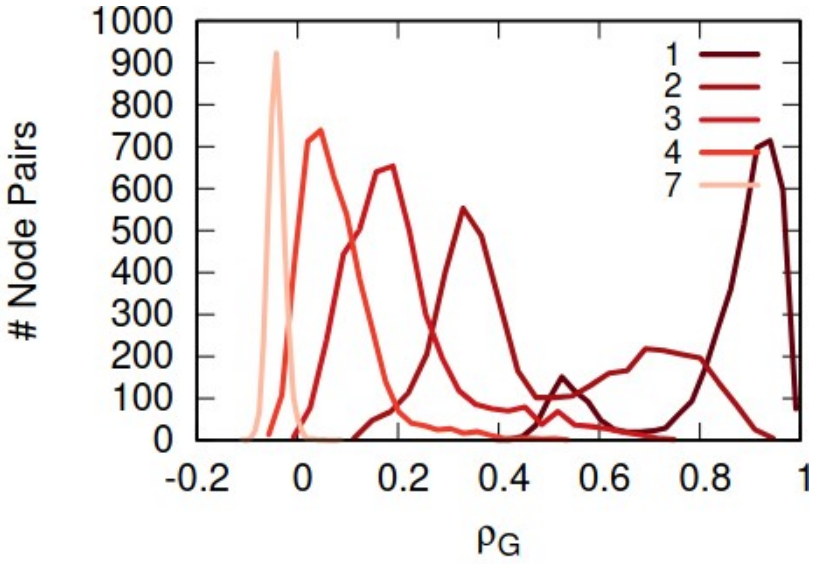
# Validation: Diffusion Processes



# Validation: SPL

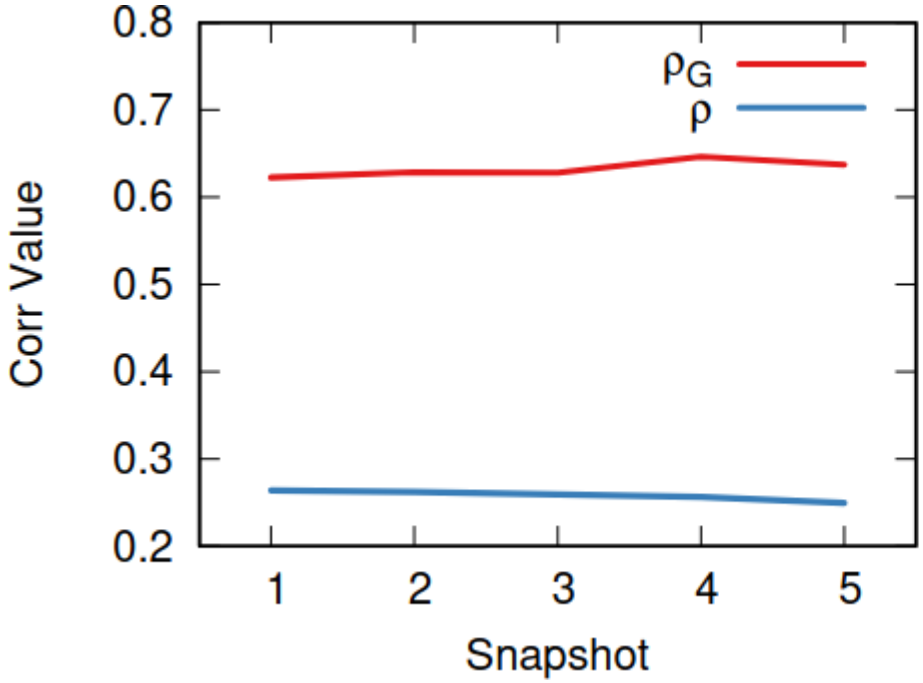



(a) $G_{n,m}$

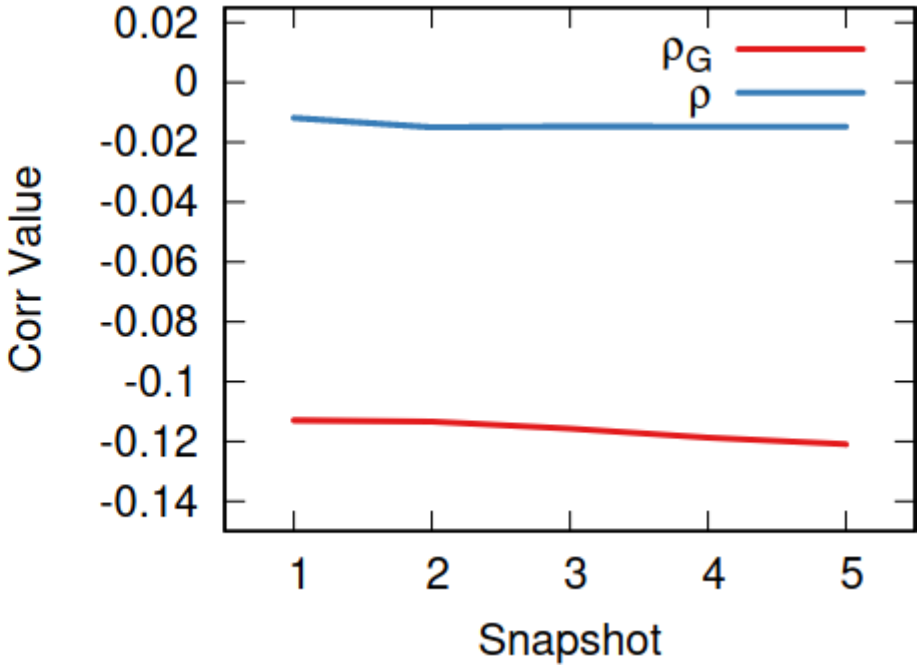



(b) Small World.

# Application




**Shelf size vs Wishlist size**




**Age vs # Tags used**

# Summary

- Pearson is useful but ignores networks
- We can plug a network in
- The resulting correlation is meaningful
- It has practical applications

# Thank you!

# Questions?

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